Analytical derivation of critical exponents of the dynamic phase transition in the mean-field approximation

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We have analyzed the dynamic phase transition of the kinetic Ising model in mean-field approximation by means of an analytical approach. Specifically, we study the evolution of the system under the simultaneous influence of time-dependent and time-independent magnetic fields. We demonstrate that within the approximate analytical treatment of our approach, the dynamic phase transition exhibits power-law dependencies for the order parameter that have the same critical exponents as the mean-field equilibrium case. Moreover we have obtained an equation of state, with which we can prove that the time-independent field component is effectively the conjugate field of the order parameter. Our analysis is limited to the parameter range, in which only second-order phase transitions occur, i.e., for small applied field amplitudes and temperatures close to the Curie point. In order to ensure the reliability of our analytical results we have corroborated them by comparison to numerical evaluations of the same model.

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I. INTRODUCTION

A ferromagnetic system under a time-dependent magnetic field can exhibit a dynamic phase transition (DPT) associated to the delayed response of the spins. The relaxation delay of the magnetization opens up the possibility that the spins can or cannot follow a time-dependent driving field [1,2]. The delay depends on the competition between two time scales, namely, the period P of the driving oscillatory field and the relaxation time τ of the magnetization itself [1]. If the magnetization can (cannot) follow the external field, a symmetric (nonsymmetric) phase is observed. The symmetric and nonsymmetric phases are also referred to as the dynamic paramagnetic and ferromagnetic phases, respectively.

In the seminal work of Tomé and Oliveira [2], the phenomenon of the dynamic phase transition was studied in the mean-field approximation of the kinetic Ising model. They showed that the order parameter Q, defined as the mean magnetization in a full field oscillation cycle, vanishes depending on the temperature and amplitude of the external field for a given field frequency. Also they identified a tricritical point that separates continuous or second-order phase transitions from discontinuous first-order phase transitions. In the discontinuous phase transition, which appears at low temperatures and high applied field amplitudes, the order parameter goes to zero discontinuously and there is a parameter range, in which symmetric and nonsymmetric solutions are both stable or metastable. However, the origin of this discontinuous phase transition is under controversy due to later results, which indicate that it may be an artifact of the mean-field approach used in these initial calculations [3,4]. On the other hand, for small field amplitudes and temperatures near the Curie point only continuous phase transitions are expected. In this case, the order parameter goes to zero continuously, as can be seen in Fig. 1.

Given the scientific interest in dynamically ordered phases and related phase transitions, many theoretical studies have been performed, primarily based on the mean-field approach [5–8] and Monte Carlo simulations [9–17]. Hereby, the author in Ref. [5] considers the dynamics of a soft-spin Ising magnet modeled by means of the time-dependent Ginzburg-Landau equation. The imaginary and real parts of the susceptibility near the dynamic transition point are studied in Ref. [6]. In Ref. [7], analytical relations were obtained for the hysteresis loop area, order parameter, and dynamic correlations. The dynamic behavior of Heisenberg magnets was also studied by several authors [10,14]. A stochastic resonance manifestation of the dynamic transition was observed in Ref. [11]. Fluctuations of the order parameter and specific heat as a function of the temperature were studied near the dynamic phase transition in Ref. [12]. The universality aspects of the kinetic Ising model have also been studied via dynamic Monte Carlo simulations [13,15], showing consistence with the critical exponents in the equilibrium case.

In spite of the many different aspects of the dynamic phase transition that were investigated in previous works [5–7,9–23], all these studies share a relevant common aspect. In all of them, only an oscillating external field component was considered. Thus, the possible role of an additionally applied time-independent field, referred to as the bias field H_b, and its possible role as the conjugate field to the order parameter Q has been treated only very recently. Monte Carlo numerical simulations [24], experimental observations in Co/Pt multilayers [25], and numerical results based on the mean-field approach [8] strongly suggest that H_b is at least a significant part of the conjugate field of the order parameter Q. While the rigorous identification of H_b as the conjugate field to Q would be in itself a valuable contribution, it has additional far-reaching consequences. Without having a well-established conjugate field, investigations of the dynamic phase transition

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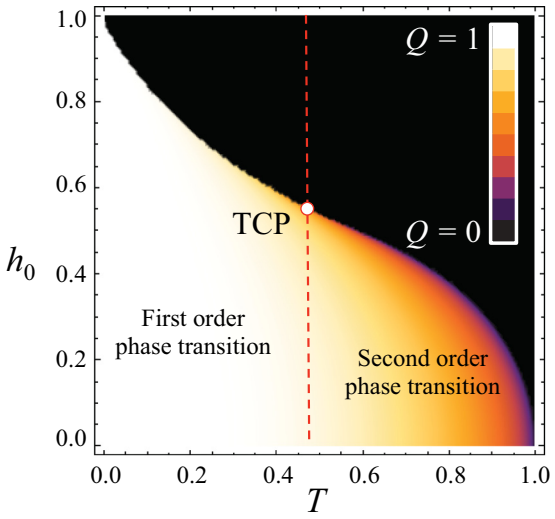

FIG. 1. (Color online) Phase diagram for the order parameter Q as a function on T and h_0 for $P = 7\tau$ and $\tau = 1$. At small fields and high temperatures the order parameter goes to zero continuously, while for high fields and low temperatures, it goes to zero discontinuously. The point that joins the two zones is the tricritical point (TCP).

are limited to the one-dimensional exploration of the external parameter space, namely, by varying the period P only, once temperature and oscillating field amplitude are chosen. This aspect is hereby equally true for theoretical, numerical, or experimental studies. If, however, the conjugate field h^* can be identified, then studies of the DPT can be extended to the two-dimensional critical regime, which is surrounding the critical point in the $P - h^*$-plane. Such two-dimensional analyses allow for far more precise and robust identifications of critical phenomena, as is well known from equilibrium studies of magnetic phase transitions (see, for instance, Ref. [26]).

In this work we present analytically derived results, which allow us to demonstrate that in the framework of the mean-field approximation, the DPT exhibits a power-law dependence of the order parameter with the same critical exponents as the mean-field equilibrium case. From our results we also extract an equation of state that clearly identifies H_b as the conjugate field to the order parameter Q. The analytical results have been verified by comparing them to numerical calculations based on the same mean-field approach.

The outline of the paper is as follows. In Sec. II, we present the mean-field dynamic equation, upon which our theory is based. From this dynamic equation, we derive by means of a series expansion analytical expressions for the order parameter Q as a function of the period P and the bias field H_b, which show critical behavior in the form of power laws. In Sec. II, we also describe the numerical method that is being used for the verification of our analytical results. In Sec. III we compare the analytical and numerical results, and in Sec. IV final remarks are presented. Finally, in the Appendix we discuss the small numerical differences between the analytical and numerical results with respect to the exact values of the critical period P_c (the period for which Q starts to deviate from zero at $H_b = 0$) and verify that they are only due to the finite lattice size used for our numerical evaluations.

II. MODEL CALCULATIONS

For a magnetic system with N spins under the influence of a time-dependent field, the mean-field Hamiltonian of the kinetic Ising model can be written as [2]

$$\mathcal{H} = -\frac{J}{N}\sum_{i,j_{i<j}} S_i S_j - H(t)\sum_i S_i, \tag{1}$$

where $J > 0$ is the exchange interaction constant, defined in units of energy. In this Ising model the dimensionless spin variables S_i can only take the values ± 1. The time-dependent magnetic field $H(t)$ is composed of an oscillating part and a time-independent static part (the bias field), for which we assume here

$$H(t) = H_0 \cos(\omega t) + H_b, \tag{2}$$

where ω and H_0 are the angular frequency and the amplitude of the oscillating field, while H_b is the bias field. In our notation both H_0 and H_b have units of energy. Even if we consider here only the specific case of a sinusoidal oscillating field contribution, one should notice that our results will have applicability for systems in which the periodic time sequence of the field is different, because the specific shape of the time-dependent field does not affect the universal aspects of the DPT [13]. If the system evolves according to the Glauber stochastic dynamics [27], it is possible to show that the magnetization of the system is given by

$$\frac{2\pi\tau}{P}\frac{dm(\eta)}{d\eta} = -m(\eta) + \tanh\left[\frac{m(\eta) + h_0\cos(\eta) + h_b}{T}\right], \tag{3}$$

where $h_{(0,b)} = H_{(0,b)}/J$, $\eta = \omega t$, and T is the temperature normalized to the Curie point. The time τ corresponds to the single spin-flip relaxation time of the magnetic system [6]. The order parameter Q is defined as the mean magnetization in a full field oscillation cycle [1,2]

$$Q = \frac{1}{2\pi}\int_{\eta'}^{\eta'+2\pi} m(\eta')d\eta', \tag{4}$$

where η is taken in the steady state regime of the time evolution for $m(t)$.

Notice that by solving numerically Eqs. (3) and (4), we can evaluate the order parameter Q as a function of the temperature T and the oscillating field amplitude h_0, as shown in Fig. 1 for the zero-bias-field case and $P = 7$. Besides the general behavior of the phase diagram, we observe a tricritical point, which separates the discontinuous (first-order) transitions at low temperature from the continuous (second-order) transitions at higher temperatures. We have determined this point by means of numerical inspections. However, we have not further investigated the tricritical or low-temperature regime here, because it has been reported that this tricritical behavior is an artifact of the mean-field model [3,4]. We therefore decided to focus on the second-order phase transition regime exclusively, for which more relevant predictions for dynamic phase transitions are expected in the framework of the mean-field model.

A. Analytical method

In this section, we analytically derive the critical behavior for the two limiting cases, in which the order parameter Q goes to zero under the following conditions: $P \to P_c$; $h_b = 0$ and $P = P_c$; $h_b \to 0$. Here, P is the period of the oscillating external field, i.e., $P = 2\pi/\omega$, and the critical period P_c is defined as the period, for which Q starts to deviate from $Q = 0$ at zero-bias field. Above (below) this period the order parameter is zero (nonzero). We have limited our analysis to the parameter regime, in which only second-order phase transitions occur, i.e., the high T and low h_0 parameter range, so that we avoid the complications of first-order phase transitions and its uncertain origin as mentioned before [3,4].

In the equilibrium case, i.e., an Ising system without oscillating magnetic field, the magnetization can be regarded as a small quantity in the vicinity of the Curie temperature. Thus, making $dm/dt = 0$ in Eq. (3) and considering $m \to 0$ near to the transition point allows for an accurate analysis of the phase transition by means of a series expansion up to third order in m. In this way, one can obtain in a straightforward fashion the well-known critical exponents $\beta = 0.5$ and $\delta = 3$ for the equilibrium mean-field theory. In the dynamic system, however, the magnetization will oscillate in time with amplitudes that are not necessarily small, because it is only the dynamic order parameter Q, which will be small in the vicinity of the dynamic phase transition. Therefore, it is not trivial to derive analytical expressions for the dynamic phase transition and the corresponding order parameter behavior and associated power laws. Instead, we need to consider some approximations, which are discussed in the following.

For the purpose of our analysis, we write the magnetization as

$$m(\eta) = Q + \Xi(\eta), \tag{5}$$

with a time-periodic term $\Xi(\eta)$ and the nonperiodic term, given by the order parameter Q. Inserting Eq. (5) into Eq. (3), we obtain

$$\frac{2\pi\tau}{P}\frac{d\Xi(\eta)}{d\eta} + Q + \Xi(\eta) = \tanh\left[\frac{Q + \Xi(\eta) + h_0\cos(\eta) + h_b}{T}\right]. \tag{6}$$

Close to critical point the order parameter is a small quantity, therefore we expand the right-hand side of Eq. (6) up to third order with respect to Q and to first order in h_b, which gives us

$$\begin{aligned}\frac{2\pi\tau}{P}\frac{d\Xi(\eta)}{d\eta} = &-Q - \Xi(\eta) + \tanh[X(\eta)] + \operatorname{sech}^2[X(\eta)]\frac{h_b}{T} \\ &+ \left\{\operatorname{sech}^2[X(\eta)] - 2\operatorname{sech}^2[X(\eta)]\tanh[X(\eta)]\frac{h_b}{T}\right\}\frac{Q}{T} \\ &+ \left\{-\operatorname{sech}^2[X(\eta)]\tanh[X(\eta)] + \left(-2 + \cosh[2X(\eta)]\operatorname{sech}^4[X(\eta)]\right)\frac{h_b}{T}\right\}\frac{Q^2}{T^2} \\ &+ \left\{-2 + \cosh[2X(\eta)]\operatorname{sech}^4[X(\eta)] - \operatorname{sech}^5[X(\eta)](-11\ \sinh[X(\eta)] + \sinh[3X(\eta)])\frac{h_b}{T}\right\}\frac{Q^3}{3T^3},\end{aligned} \tag{7}$$

where we have defined the periodic function

$$X(\eta) = \frac{\Xi(\eta) + h_0\cos(\eta)}{T}. \tag{8}$$

One should notice that the previous expansion can be justified since we want to analyze the dynamic behavior of the order parameter in the case $P \to P_c$, which implicates that Q is small. We also consider h_b to be a small quantity, because the critical point is associated with $h_b = 0$, so that h_b can be made arbitrarily small for the exploration of the critical regime. After integrating Eq. (7) for a period of length P, we obtain the following equation:

$$\frac{Q^3}{3T^2} = \left(\frac{I_1(P) - 2\pi T}{I_2(P)}\right)Q + \left(\frac{I_1(P)}{I_2(P)} + \frac{Q^2}{T^2}\right)h_b, \tag{9}$$

where the integrals $I_i(P)$ in Eq. (9) are defined as

$$I_1(P) = \int_0^{2\pi} \operatorname{sech}^2[X(\eta)]\,d\eta \tag{10}$$

and

$$I_2(P) = \int_0^{2\pi} (\cosh[2X(\eta)] - 2)\operatorname{sech}^4[X(\eta)]\,d\eta. \tag{11}$$

In Eq. (9), we can neglect the term $Q^2 h_b$, because in the case $h_b \to 0$ the order parameter goes to zero at P_c, while I_1/I_2 does not, which can be seen in Fig. 2. Therefore we arrive to

$$h_b = \frac{2\pi T - I_1(P)}{I_1(P)}Q + \frac{I_2(P)}{3T^2 I_1(P)}Q^3. \tag{12}$$

With Eq. (12), we have obtained the so-called *state equation*, which connects the key quantities of our system, namely, the order parameter Q, the period P, and the nonoscillating external field h_b for every choice of T and h_0.

In Eqs. (10) and (11) we have highlighted the P dependence of I_1 and I_2 since $\Xi(\eta)$ depends on P, although they also depend on the other relevant parameters such as h_0, T, etc. It should also be mentioned that Eq. (12) has exactly the same form as the equation of state for the equilibrium mean-field model, if one replaces h_b with h and Q with m.

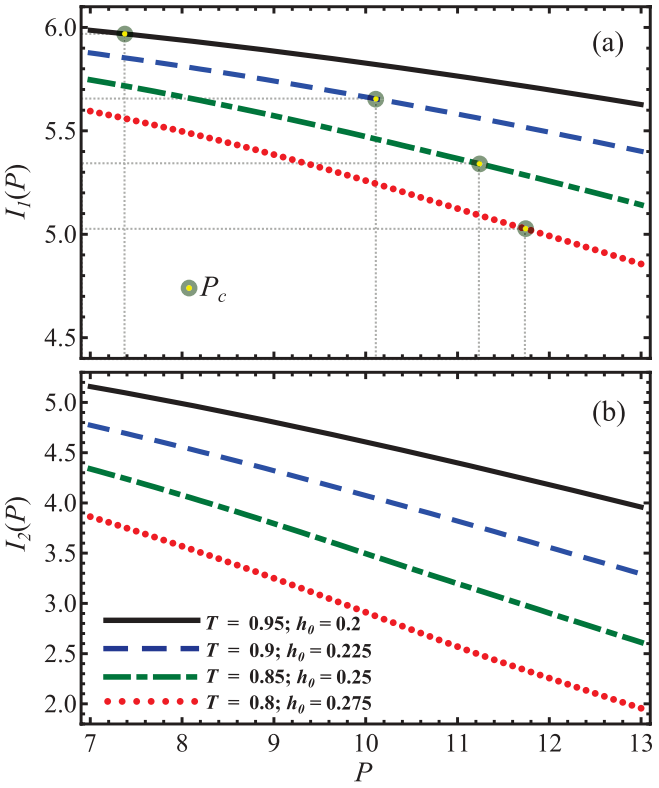

FIG. 2. (Color online) Integrals (a) I_1 and (b) I_2 defined in Eqs. (10) and (11) as a function of P for several combinations of T and h_0. The respective T and h_0 values for both plots are given by the legend in (b). In (a) all curves have a highlighted dot, which is the point that is associated with the critical period P_c condition, Eq. (14). We consider $\tau = 1$ everywhere.

This equivalency can be considered to be the formal proof that h_b is the conjugate field to Q, just as h is the conjugate field to m.

In the case $h_b = 0$, Eq. (12) reduces to $Q = 0$ or

$$Q^2 = 3T^2\left(\frac{I_1(P) - 2\pi T}{I_2(P)}\right). \tag{13}$$

Equation (13) allows us to derive the critical point condition, because when $P = P_c$, the order parameter Q is zero and therefore, the critical point is defined by

$$I_1(P = P_c) = 2\pi T. \tag{14}$$

One should notice that in the absence of an oscillating applied field ($h_0 = 0$), the periodic function $\Xi(\eta)$ is zero, and according to Eq. (10), $I_1 = 2\pi$. Correspondingly, Eq. (14) establishes that the normalized temperature $T = 1$ is a critical point, a result that reflects exactly the equilibrium case. In others words, we have derived a critical condition for the dynamic phase transition that also applies to the equilibrium case, where the Curie temperature gives the critical point, at which the order parameter vanishes.

Now, we consider the bias field in the limit $h_b \to 0$ with $P = P_c$. Using the condition (14) in Eq. (12) we arrive at the following critical behavior for the order parameter as a function of the bias field:

$$Q = T\left(\frac{6\pi}{I_2(P_c)}\right)^{1/3} h_b^{1/3}. \tag{15}$$

This formal equivalency to the equilibrium case corroborates the bias field as the *conjugate field*, and also results in a critical exponent $\delta = 3$ that is identical to the exponent obtained in mean-field approximation for the equilibrium system. The relevant aspect here is that this result is valid for any periodic function $\Xi(\eta)$ and therefore underscores the generality of our claim.

With respect to Eq. (13), one needs to carefully analyze the behavior of the integrals $I_i(P)$ as a function of the period P. In the approximation (7) we assume that we are close to the critical period P_c, because in this case we can consider Q as a small quantity, but this condition is not explicit in the definitions (10) and (11). Thus, we need to evaluate the integrals $I_i(P)$ for P in the vicinity of P_c. In Fig. 2, we solved numerically the integrals $I_1(P)$ and $I_2(P)$ for several values of T and h_0. The dots in Fig. 2 are representations of the critical condition Eq. (14) and thus, they indicate the critical period. As is shown in this figure, the behavior of the integrals around the critical period is that of a regular function, meaning that they do not show any critical behavior themselves. This regular behavior of the integrals can be understood because they are defined according to the Taylor expansion done to derive Eq. (7), which implies that they are evaluated near $Q = 0$ and therefore they do not have irregularities at $P = P_c$. Based on this idea, we are going to expand the integrals (10) and (11) for P close to the critical period, namely, we expand the right-hand side of Eq. (13) near $P = P_c$, and we obtain

$$Q = T\left(\frac{3|\partial_P I_1(P)|_{P=P_c}}{I_2(P_c)}\right)^{1/2}(P_c - P)^{1/2}. \tag{16}$$

Equation (16) allows us to establish the critical behavior $Q(P \to P_c; h_b = 0) \propto (P_c - P)^{\beta}$ with the critical exponent $\beta = 0.5$, which also coincides with the critical exponent obtained in the mean-field approximation at equilibrium [28].

The equation of state, Eq. (12), its implications, and the critical behavior given by relations (15) and (16) are the main results presented in this paper. They establish the identity of the conjugate field to the order parameter Q and demonstrate that the critical exponents of Q have the same values that apply to the magnetization vs temperature and field relations in the equilibrium case.

B. Numerical method

As mentioned before, the analytical results have been corroborated by numerical calculations based also on the mean-field approach [Eq. (1)]. Even though details of our calculations have been described previously in Ref. [8], we report key aspects here again for transparency.

By means of discretizing the differential equation, Eq. (3), into K points we obtain

$$m(k) = F[m(k)] = -\frac{K}{2}\frac{\omega\tau}{2\pi}[m(k+1) - m(k-1)] + \tanh\left\{\frac{m(k) + h_0\sin(2\pi k/K) + h_b}{T}\right\}, \tag{17}$$

as a system of coupled nonlinear algebraic equations, where k is the discretization index running from 1 to K to represent a complete oscillation cycle within the calculation. The order parameter Q is then given as the magnetization average over all k points:

$$Q = \frac{1}{K}\sum_{k=1}^{K} m(k). \tag{18}$$

Contrary to analytical calculations where the approach to the dynamically stable state solution is described by the actual dynamic solution of the differential equation, the discretized equation system (17) requires an iterative process to achieve a steady-state solution. In order to minimize the calculation time, we have chosen the magnetization $m(k)$ for the first iteration step to be initialized as a sinusoidal function with the same period as the applied field and in phase with it plus a continuous magnetization term, namely,

$$m_{i=1} = 0.15 + 0.4\,\sin\left(2\pi\frac{k}{K}\right). \tag{19}$$

For iteration $i+1$, $m(k)$ is calculated from $m(k)$ for iteration i by evaluating

$$m_{i+1}(k) = m_i(k) + s\,\{F\,[m_i(k)] - m_i(k)\}\,, \tag{20}$$

whereby $F\,[m(k)]$ for every i is determined by utilizing Eq. (17). In order to keep the iteration procedure stable we have chosen $s < 1$ and specifically implemented $s = 0.5$ in our iteration scheme. We consider the numerical solution of the dynamically stable state achieved, when the cut-off condition, which is defined as

$$\max\{F\,[m_i(k)] - m_i(k)\} < 10^{-10},$$

is fulfilled. This means that the maximum difference between the $F\,[m(k)]$ and $m_i(k)$ is lower than 10^{-10} for all $k = 1$ to K. If not stated otherwise, K was set to 200 for our numerical calculations. Numerical evaluations for larger K were also performed, as discussed in the Appendix, but did not result in any relevant difference, except for the absolute value of the critical period P_c, which is K dependent.

III. RESULTS AND DISCUSSION

In this section we compare the analytically derived critical behavior given by Eqs. (15) and (16), with the numerical results. First, we analyze the behavior of the order parameter vs P/P_c in the case $h_b = 0$, i.e., Eq. (16). Second, we check the critical behavior given by Eq. (15), i.e., we discuss the case $P = P_c$ when a small external field h_b is applied to the system.

The behavior in the case $Q(P \to P_c; h_b = 0)$ vs P/P_c [29] is displayed in Fig. 3, where several curves are shown for different h_0 and T parameters calculated using Eq. (16) (solid lines) and obtained by numerical evaluations (dots). As expected, for lower T and higher h_0 one begins to see some small discrepancies between both methods, but overall, there is very good agreement between the analytical and numerical results, thus validating our analytical approximations. Hereby, it is important to highlight that in all of the analytic results presented here, we have not used any fitting or calibration factor.

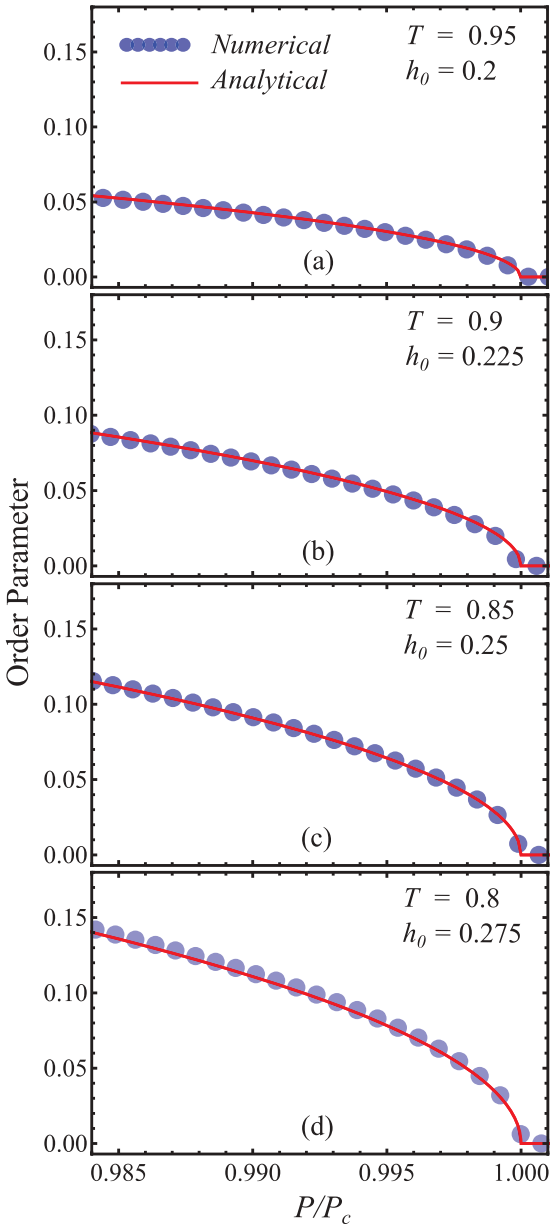

FIG. 3. (Color online) Order parameter as a function of P/P_c. The solid lines represent analytical results and the dots are numerical results. From (a) to (d) the temperature changes from 0.95 to 0.8, illustrating that the quality of our analytical approach degrades with decreasing temperature. We consider $\tau = 1$ everywhere.

The small discrepancies can be explained as follows. At lower temperatures, the order parameter Q is higher for the same P/P_c ratio (see Fig. 3), therefore our approximation of small Q used to derive Eq. (7) begins to lose its validity and the analytical approach is less precise. Thus, we do not expect that our results can be applied in the entire temperature range in the same way, because the resulting Q values change substantially for identical P/P_c ratios, as can be observed in Fig. 3 for the different temperatures. But despite these limits, the analytical expression has a substantial reliability range.

In Fig. 4 we show the case $Q(P = P_c; h_b \to 0)$, where we compare the results from Eq. (15) with the numerical solutions. Also here, one can see that the analytical critical

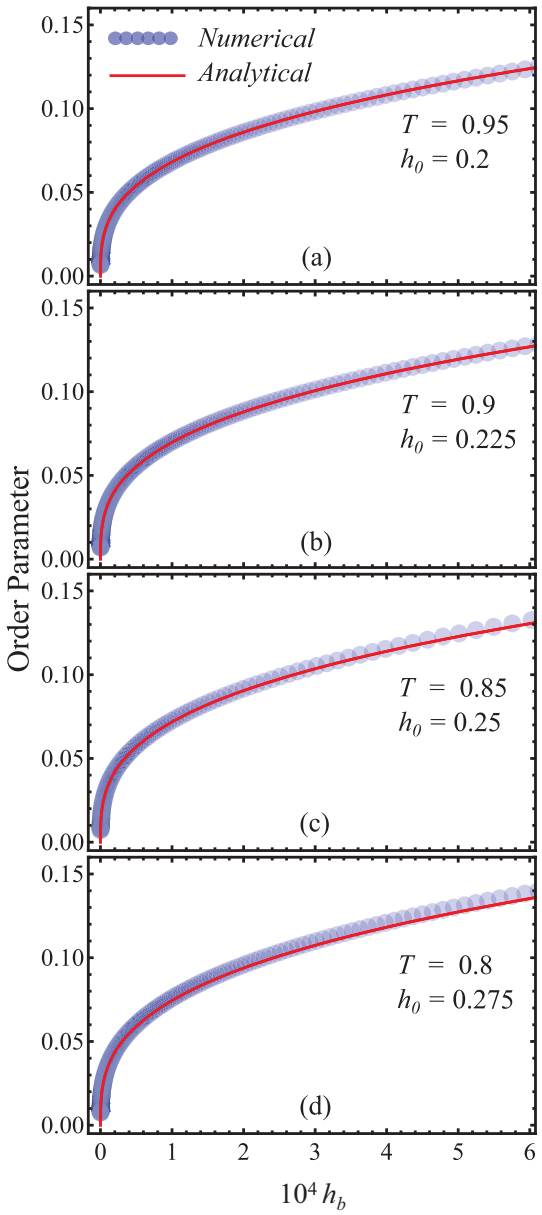

FIG. 4. (Color online) Order parameter as a function of the bias field. The solid lines are the analytical results from Eq. (15) and the dots correspond to the numerical results. The temperature decreases from (a) 0.95 to (d) 0.8, showing that, for the lower temperatures [(c) and (d)], the analytical curve starts to deviate from the numerical one as the bias field increases.

behavior obtained in Sec. II A is in very good agreement with the numerical results, and therefore corroborates the validity of our analytical approach. But despite this very good quantitative agreement, it is also evident from Fig. 4 that the precision deteriorates in terms of the applicable h_b range, as one moves to lower temperatures.

IV. FINAL REMARKS

We have developed an analytical method to study the dynamic phase transition in an Ising ferromagnet in mean-field approximation. We have obtained an analytic state equation [Eq. (12)], which allowed us to demonstrate the power-law behavior of the order parameter as a function of the bias field [Eq. (15)] and the period [Eq. (16)]. We have also shown that the bias field plays the role of the conjugate field here, as well as exhibits the same critical exponent ($\delta = 3$) as in the mean-field equilibrium case.

We have compared the analytical results with precise numerical evaluations, and we have found very good agreement between both approaches. Therefore, we can conclude that the theory and approximations presented in this work are valid and useful in a considerable temperature range close to the Curie point, where second-order dynamic phase transitions occur. The reason for the observed agreement between the analytical and numerical approach is of course the fact that both methods are derived from the same starting point, namely, the dynamic equation (3). However, in contrast to any direct numerical solution of Eq. (3), the analytical method allows us to derive the universal aspects of the critical behavior in closed form and furthermore, enables us to derive the state equation, Eq. (12). However, it is clear that further results based on more advanced (non-mean-field) models, such as dynamic Monte Carlo simulations, for instance, will be most helpful to obtain a more detailed understanding of the critical behavior of dynamic phase transitions. It would be most interesting to demonstrate, if the here obtained results, especially the existence of a state equation could be verified by such advanced method, and it is our hope that this work will stimulate exactly these types of investigations.

While previous numerical results suggested the same critical exponents as for the equilibrium mean-field theory, our analytical treatment can be considered a real proof since we have derived the critical exponents $\beta = 0.5$ and $\delta = 3$ in a formally closed analytic form. Therefore, our theory identifies the universal aspects of the second-order dynamic phase transition in mean-field Ising ferromagnets in the presence of both an oscillating external field and a static bias field.

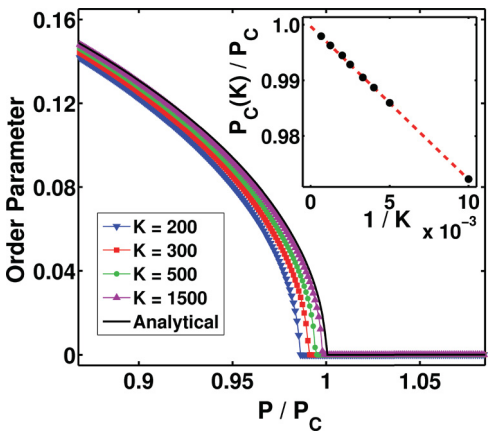

FIG. 5. (Color online) Order parameter as a function of the period normalized to the analytical critical period for numerical evaluations of different lattice sizes. The agreement between analytical and numerical results continuously improves for increasing K. The inset shows an extrapolation for the numerically determined $P_c(K)$ values in the limit of $1/K \to 0$.

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APPENDIX

As mentioned in Secs. II B and III, the numerically extracted critical period P_c depends on the lattice size K, and thus differs from the corresponding P_c values that are obtained analytically. Thus, in order to verify that this difference is due to the finite lattice size K only, we calculated Q versus P for different K values by means of numerical evaluations for $h_0 = 0.25$ and $T = 0.95$. In Fig. 5 we show the influence of the lattice size on P_c by representing the order parameter versus period normalized to the critical period calculated from the analytical solution P_c. One can observe that the $P_c(K)$ values extracted from the numerical calculations approach the analytical P_c value as K increases. The inset of Fig. 5 shows the critical period $P_c(K)$ extracted from the Q vs P data sets and normalized to the analytical value of P_c as a function of $1/K$. Using a linear approximation, shown as a dashed line in the inset of Fig. 5, we find that the extrapolated $P_c(K)/P_c$ value is 0.999 ± 0.001 for $K \to \infty$, which confirms that the P_c differences in between our numerical calculations and the analytical formulation are solely due to finite lattice size effects.

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[28] Notice that, to evaluate $\partial_P I_1(P)$, we use the fact that $I_1(P)$ varies linearly with the period when P is close to the critical period, and therefore we can evaluate $\partial_P I_1(P)$ as $\Delta I_1(P)/\Delta P$ with ΔP taken in a range $P < P_c$. Also, the slope in $I_1(P)$ vs P is practically a constant close to the critical period, but it is worth mentioning that it is formally correct to take the derivative $\partial_P I_1(P)$ for $P = P_c - \delta'$, with $\delta' \to 0$, and $\delta' > 0$.
[29] The reason why we normalized the period P to the critical value P_c is to avoid the discrepancy of P_c values in between both methods. As shown in the Appendix, this discrepancy arises due to the discretization of the differential equation using a finite lattice of size K.